

A simple proof on the inequality of arithmetic and geometric means

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Abstract

In this short paper we show that the inequality of arithmetic and geometric means is reduced to another interesting inequality, and a proof is provided.

1 Introduction

Given n arbitrary real numbers a_1, a_2, \dots, a_n , we define their arithmetic and geometric means as following:

Definition 1.1. The *Arithmetic Mean* is:

$$AM(a_1, a_2, \dots, a_n) = \frac{a_1 + a_2 + \dots + a_n}{n} \quad (1.1)$$

Definition 1.2. If such n real numbers are all non-negative, the *Geometric Mean* is:

$$GM(a_1, a_2, \dots, a_n) = \sqrt[n]{a_1 \cdot a_2 \cdots a_n} \quad (1.2)$$

The inequality of arithmetic and geometric means states that the arithmetic mean is greater than or equal to the geometric mean if those real numbers are all *positive*:

Theorem 1.3. For arbitrary n positive real numbers a_1, a_2, \dots, a_n , the inequality

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdots a_n} \quad (1.3)$$

holds, with equality if and only if $a_1 = a_2 = \dots = a_n$

The inequality of arithmetic and geometric means is so famous that there are various proofs in the literature [1, 2, 3, 4, 5, 6]. In this short paper we provide a simple proof which uses another interesting inequality.

2 Proof

If we use the following notations:

$$x = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$d_i = a_i - x$$

We see that $x > 0, x + d_i = a_i > 0, \sum_{i=1}^n d_i = 0$ and $a_1 = a_2 = \cdots = a_n$ implies that $d_1 = d_2 = \cdots = d_n = 0$. Then Theorem 1.3 is actually reduced to the following inequality:

Theorem 2.1. Let x be a positive real number, d_1, d_2, \dots, d_n be real numbers that each $d_i > -x$ and $\sum_{i=1}^n d_i = 0$, then

$$x \geq \sqrt[n]{(x + d_1) \cdot (x + d_2) \cdots (x + d_n)} \quad (2.1)$$

$$x^n \geq (x + d_1) \cdot (x + d_2) \cdots (x + d_n) \quad (2.2)$$

Both equalities hold if and only if $d_1 = d_2 = \cdots = d_n = 0$.

Proof. Since x and $x + d_i$ are positive real numbers, (2.1) is equivalent to (2.2). We prove the second inequality using induction on n .

1. When $n = 1$ and 2, it is easy to verify the correctness.
2. Suppose that when $n = k (\geq 2)$, the inequality is true. That is

$$x^k \geq (x + d_1) \cdot (x + d_2) \cdots (x + d_k)$$

3. Now assume $n = k + 1$. If $d_1 = d_2 = \cdots = d_{k+1} = 0$, the equality is trivially true. Otherwise suppose that d_1, d_2, \dots, d_{k+1} are not all zero. Then there must be one $d_u > 0$, one $d_v < 0$, and $u \neq v$ since $n > 2$. Without loss of generality, assume that $d_{k+1} > 0$ and $d_k < 0$. Since $d_k > -x$, it is obviously true that $d_k + d_{k+1} > -x$. It is clear that $x, d_1, d_2, \dots, d_{k-1}, (d_k + d_{k+1})$ also meet the prerequisites of the inequality, therefore

$$x^k \geq (x + d_1) \cdot (x + d_2) \cdots (x + (d_k + d_{k+1}))$$

And we get

$$\begin{aligned} x^{k+1} &\geq (x + d_1) \cdot (x + d_2) \cdots (x + (d_k + d_{k+1})) \cdot x \\ &= (x + d_1) \cdot (x + d_2) \cdots (x^2 + (d_k + d_{k+1}) \cdot x) \\ &> (x + d_1) \cdot (x + d_2) \cdots (x^2 + (d_k + d_{k+1}) \cdot x + d_k \cdot d_{k+1}) \quad \text{since } d_k \cdot d_{k+1} < 0 \\ &= (x + d_1) \cdot (x + d_2) \cdots (x + d_k) \cdot (x + d_{k+1}) \quad \blacksquare \end{aligned}$$

Now we complete the proof, and the equality holds if and only if $d_1 = d_2 = \cdots = d_n = 0$. \square

References

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